$\boxed{1(a)}$ Solve $y'' + 3y' + 2y = 6$.

Step 1: Solve homogeneous equation $y'' + 3y' + 2y = 0$.

The characteristic equation is $r^2 + 3r + 2 = 0$

which has roots:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 \pm \sqrt{1}}{2} = \frac{-3+1}{2}, \frac{-3-1}{2}$$

$$= -1, -2$$

Thus, $\boxed{y_h = c_1 e^{-x} + c_2 e^{-2x}}$

Step 2: Find a particular solution y_p to

$y'' + 3y' + 2y = 6$.

Guess $y_p = A$.

Note this does not appear in y_h so we are good to continue

Then, $y_p' = 0$

$y_p'' = 0$.

Plug these into $y'' + 3y' + 2y = 6$ to get $2A = 6$.

So, $A = 3$.

Thus, $\boxed{y_p = 3}$.

Step 3: The general solution is

$$\boxed{y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3}$$

$\boxed{①(b)}$ Solve $y'' - 10y' + 25y = 30x + 3$

Step 1: Solve homogeneous eqn. $y'' - 10y' + 25y = 0$.

The characteristic equation is $r^2 - 10r + 25 = 0$ which has roots

$$r = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)} = \frac{10 \pm \sqrt{0}}{2} = 5$$

Thus,

$$y_h = c_1 e^{5x} + c_2 x e^{5x}$$

Step 2: Find a particular solution y_p to $y'' - 10y' + 25y = 30x + 3$.

Guess: $y_p = Ax + B$.

Note this does not appear in y_h so we are good to continue

So,
$$y_p' = A$$
$$y_p'' = 0$$

Plug this into $y'' - 10y' + 25y = 30x + 3$ to get

$$-10A + 25(Ax + B) = 30x + 3.$$

Regroup left-hand side to get

$$\underbrace{25A}\,x + \underbrace{(-10A + 25B)} = \underbrace{30}\,x + 3.$$

This gives the system

$$\boxed{\begin{aligned} 25A \quad\quad &= 30 \\ -10A + 25B &= 3 \end{aligned}}$$

Thus, $A = \frac{30}{25} = \frac{6}{5}$.

And $B = \frac{3}{25} + \frac{10}{25}A = \frac{3}{25} + \frac{10}{25} \cdot \frac{6}{5} = \frac{15}{25} = \frac{3}{5}$

So, $\boxed{y_p = Ax + B = \frac{6}{5}x + \frac{3}{5}}$

<u>Step 3:</u> The general solution to $y'' - 10y' + 25y = 30x + 3$ is thus

$$\boxed{y = y_h + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}}$$

$\boxed{1(c)}$ Solve $\frac{1}{4}y'' + y' + y = x^2 - 2x$.

Step 1: Solve the homogeneous eqn. $\frac{1}{4}y'' + y' + y = 0$
which has characteristic eqn. $\frac{1}{4}r^2 + r + 1 = 0$
The roots are

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1/4)(1)}}{2(\frac{1}{4})} = \frac{-1 \pm \sqrt{0}}{(1/2)} = \frac{-1}{1/2} = -2$$

Thus,

$$\boxed{y_h = c_1 e^{-2x} + c_2 x e^{-2x}}$$

Step 2: Find a particular solution to
$\frac{1}{4}y'' + y' + y = x^2 - 2x$.

Guess: $y_p = Ax^2 + Bx + C$ ← This doesn't appear in y_h so we are good to continue

We get $y_p' = 2Ax + B$
$y_p'' = 2A$

Plug this into $\frac{1}{4}y'' + y' + y = x^2 - 2x$ to get

$$\frac{1}{4}(2A) + (2Ax + B) + (Ax^2 + Bx + C) = x^2 - 2x$$

Rearrange the left-hand side to get

$$A x^2 + (2A+B)x + \left(\tfrac{1}{2}A + B + C\right) = x^2 - 2x + 0$$

So we get

$$\boxed{\begin{aligned} A &= 1 \quad (1)\\ 2A+B &= -2 \quad (2)\\ \tfrac{1}{2}A+B+C &= 0 \quad (3)\end{aligned}}$$

(1) gives $A = 1$.

(2) gives $B = -2 - 2A = -2 - 2(1) = -4$

(3) gives $C = -\frac{1}{2}A - B = -\frac{1}{2}(1) - (-4) = 4 - \frac{1}{2} = \frac{7}{2}$

Thus,

$$\boxed{y_p = Ax^2 + Bx + C = x^2 - 4x + \frac{7}{2}}$$

Step 3: The general solution to $\frac{1}{4}y'' + y' + y = x^2 - 2x$ is

$$\boxed{y = y_h + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}}$$

① (d) Solve $y'' + 3y = xe^{3x}$

Step 1: Solve the homogeneous eqn. $y'' + 3y = 0$.
The characteristic equation is $r^2 + 3 = 0$
which has roots

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(3)}}{2(1)} = \frac{\pm\sqrt{-12}}{2} = \frac{\pm\sqrt{4}\sqrt{-3}}{2}$$

$$= \frac{\pm 2\sqrt{-3}}{2} = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1} = \underbrace{\pm\sqrt{3}\,i}_{0 \pm \sqrt{3}\,i}$$

Thus,

$$\boxed{\begin{aligned} y_h &= c_1 e^{0x}\cos(\sqrt{3}x) + c_2 e^{0x}\sin(\sqrt{3}x) \\ &= c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x) \end{aligned}}$$

Step 2: Find a particular solution y_p to
$y'' + 3y = 24xe^{3x}$

Guess: $y_p = (Ax + B)e^{3x}$ ← not part of y_h so we are good to continue

So, $y_p = Axe^{3x} + Be^{3x}$

$y_p' = Ae^{3x} + 3Axe^{3x} + 3Be^{3x}$

$y_p'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} + 9Be^{3x}$

Plugging these into $y'' + 3y = xe^{3x}$ gives

$$\underbrace{(6A+9B)e^{3x} + 9Axe^{3x}}_{y_p''} + \underbrace{3Axe^{3x} + 3Be^{3x}}_{3y_p} = xe^{3x}$$

Combining like terms gives

$$\underbrace{(6A+12B)}_{0}e^{3x} + (12A)xe^{3x} = xe^{3x}$$

So we get

$$\boxed{\begin{aligned} 12A &= 1 \quad (1) \\ 6A + 12B &= 0 \quad (2) \end{aligned}}$$

(1) gives $A = 1/12$

(2) gives $B = \frac{-6}{12}A = -\frac{1}{2}\left(\frac{1}{12}\right) = -\frac{1}{24}$

Thus, $\boxed{y_p = (Ax+B)e^{3x} = \left(\frac{1}{12}x - \frac{1}{24}\right)e^{3x}}$

Step 3: The general solution to $y'' + 3y = xe^{3x}$ is

$$\boxed{y = y_h + y_p = c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x) + \left(\frac{1}{12}x - \frac{1}{24}\right)e^{3x}}$$

① (e) Solve $4y'' - 4y' - 3y = \cos(2x)$

Step 1: Solve the homogeneous eqn. $4y'' - 4y' - 3y = 0$

The characteristic eqn is $4r^2 - 4r - 3 = 0$.

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-3)}}{2(4)} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8}$$

$$= \frac{4+8}{8}, \frac{4-8}{8} = \frac{3}{2}, -\frac{1}{2}$$

Thus,

$$\boxed{y_h = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}}$$

Step 2: Find a particular solution y_p to

$$4y'' - 4y' - 3y = \cos(2x).$$

Guess: $y_p = A\cos(2x) + B\sin(2x)$ ← this doesn't appear in y_h so we are good to continue

Then, $y_p' = -2A\sin(2x) + 2B\cos(2x)$

$y_p'' = -4A\cos(2x) - 4B\sin(2x)$

Plugging these into $4y'' - 4y' - 3y = \cos(2x)$ gives

$$-16A\cos(2x) - 16B\sin(2x) + 8A\sin(2x) - 8B\cos(2x)$$
$$-3A\cos(2x) - 3B\sin(2x) = \cos(2x).$$

Combine like terms on the left-hand side to get

$$\underbrace{(-19A-8B)}_{1}\cos(2x)+\underbrace{(8A-19B)}_{0}\sin(2x)=\cos(2x)$$

This gives

$$\boxed{\begin{aligned} -19A-8B&=1 \quad (1)\\ 8A-19B&=0 \quad (2)\end{aligned}}$$

(2) gives $A=\frac{19}{8}B$.

Plug this into (1) to get $-19\left(\frac{19}{8}B\right)-8B=1$.

This gives $\frac{-361-64}{8}B=1$.

So, $B=\frac{-8}{425}$

Thus, $A=\frac{19}{8}B=\frac{19}{8}\left(\frac{-8}{425}\right)=\frac{-19}{425}$

So, $\boxed{y_p=\frac{-19}{425}\cos(2x)-\frac{8}{425}\sin(2x)}$

Step 3: The general solution to $4y''-4y'-3y=\cos(2x)$ is

$$\boxed{y=y_h+y_p=c_1e^{\frac{3}{2}x}+c_2e^{-\frac{1}{2}x}-\frac{19}{425}\cos(2x)-\frac{8}{425}\sin(2x)}$$

2(a) Solve $y''-y'=-3$

Step 1: Solve the homogeneous eqn $y''-y'=0$.

The characteristic eqn is $r^2-r=0$

This factors as $r(r-1)=0$.

The roots are $r=0,1$.

Thus,

$$\boxed{y_h = c_1 e^{0x} + c_2 e^{x} = c_1 + c_2 e^{x}}$$

Step 2: Find a particular solution y_p to $y''-y'=-3$.

We want to guess $y_p = A$.

However, the constant solution appears in y_h

So we multiply by x and instead guess

$$y_p = Ax$$

Which doesn't occur in y_h.

Then, $y_p' = A$

$y_p'' = 0$

Plugging this into $y''-y'=-3$ gives

$$0 - A = -3$$

So, $A=3$.

Thus, $\boxed{y_p = 3x}$.

Step 3: The general solution to $y'' - y' = -3$ is

$$\boxed{y = y_h + y_p = c_1 + c_2 e^x + 3x}$$

2(b) Solve $y''-16y=2e^{4x}$

Step 1: Solve the homogeneous eqn $y''-16y=0$.

The characteristic eqn is $r^2-16=0$.

The roots are $r=\pm 4$.

Thus,

$$\boxed{y_h = c_1 e^{4x} + c_2 e^{-4x}}$$

Step 2: Find a particular solution y_p to

$y''-16y=2e^{4x}$.

We first think to guess $y_p = Ae^{4x}$.

However this appears as a term in y_h.

So we multiply by x to get $y_p = Axe^{4x}$

which doesn't appear in y_h.

Then,

$$y_p = Axe^{4x}$$
$$y_p' = Ae^{4x} + 4Axe^{4x}$$
$$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$$

Plugging this into $y''-16y=2e^{4x}$ gives

$$\underbrace{8Ae^{4x}+16Axe^{4x}}_{y_p''} - \underbrace{16Axe^{4x}}_{y_p} = 2e^{4x}$$

So we get

$$8Ae^{4x} = 2e^{4x}$$

Thus, $A = \frac{1}{4}$.

So, $\boxed{y_p = \frac{1}{4}xe^{4x}.}$

Step 3: The general solution to $y'' - 16y = 2e^{4x}$ is

$$\boxed{y = y_h + y_p = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4}xe^{4x}}$$

2(c) Solve $y'' + 2y' = 2x + 5 - e^x$

Step 1: Solve the homogeneous eqn $y'' + 2y' = 0$.

The characteristic eqn is $r^2 + 2r = 0$.

This factors as $r(r+2) = 0$.

The roots are $r = 0, -2$.

Thus,

$$\boxed{y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}}$$

Step 2: Find a particular solution y_p to

$y'' + 2y' = 2x + 5 - e^x$.

Guess: $y_p = Ax^2 + Bx + Ce^x$

$y_p' = 2Ax + B + Ce^x$

$y_p'' = 2A + Ce^x$

At first you might guess $y_p = Ax + B + Ce^x$ but y_h has a constant term so bump $Ax+B$ up by an x to get the $Ax^2 + Bx$ term

Plugging this into $y'' + 2y' = 2x + 5 - e^x$ gives

$$\underbrace{2A + Ce^x}_{y_p''} + \underbrace{4Ax + 2B + 2Ce^x}_{2y_p'} = 2x + 5 - e^x$$

Combining like terms gives

$$4Ax + (2A+2B) + 3Ce^x = 2x + 5 - e^x$$

So,

$$\boxed{\begin{aligned} 4A \quad &= 2 \quad (1)\\ 2A + 2B \quad &= 5 \quad (2)\\ 3C &= -1 \quad (3)\end{aligned}}$$

(1) gives $A = \frac{1}{2}$.

(2) gives $B = \frac{5}{2} - A = \frac{5}{2} - \frac{1}{2} = 2$

(3) gives $C = -1/3$.

Thus, $\boxed{y_p = \frac{1}{2}x^2 + 2x - \frac{1}{3}e^x}$

Step 3: The general solution to

$y'' + 2y' = 2x + 5 - e^x$ is

$$\boxed{y = y_h + y_p = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x - \frac{1}{3}e^x}$$

(2)(c) Solve $y'' + 2y' = 2x + 5 - e^{-2x}$

Step 1: Solve the homogeneous eqn $y'' + 2y' = 0$.

The characteristic eqn is $r^2 + 2r = 0$.

This factors as $r(r+2) = 0$.

The roots are $r = 0, -2$.

Thus,

$$\boxed{y_h = c_1 e^{0x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}}$$

Step 2: Find a particular solution y_p to

$y'' + 2y' = 2x + 5 - e^{-2x}$.

Guess: $y_p = Ax^2 + Bx + Cxe^{-2x}$

As in 2(c) we need the $Ax^2 + Bx$ term but also since e^{-2x} appears in y_h also we want the xe^{-2x} term in y_p

$y_p' = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$

$y_p'' = 2A - 2Ce^{-2x} - 2Ce^{-2x} + 4Cxe^{-2x}$

Plugging this into $y'' + 2y' = 2x + 5 - e^{-2x}$ we get

$$\underbrace{2A - 4Ce^{-2x} + 4Cxe^{-2x}}_{y_p''} + \underbrace{4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x}}_{2y_p'}$$

$$= 2x + 5 - e^{-2x}$$

Combining like terms on the left-hand side we get

$$4Ax + (2A+2B) - 2Ce^{-2x} = 2x + 5 - e^{-2x}$$

So,

$$\boxed{\begin{aligned} 4A &= 2 \quad (1) \\ 2A+2B &= 5 \quad (2) \\ -2C &= -1 \quad (3) \end{aligned}}$$

(1) gives $A = \frac{1}{2}$.
(2) gives $B = 5/2 - A = 5/2 - 1/2 = 2$
(3) gives $C = \frac{1}{2}$.

Thus, $\boxed{y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$

Step 3: The general solution to

$y'' + 2y' = 2x + 5 - e^{-2x}$ is

$$\boxed{y = y_h + y_p = c_1 + c_2e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}$$

③(a) Solve

$$y'' + 3y' + 2y = 6, \quad y'(0) = 0, \; y(0) = 0$$

In problem ① we saw that the general solution to $y'' + 3y' + 2y = 6$ is

$$y = c_1 e^{-x} + c_2 e^{-2x} + 3$$

We want this solution to satisfy $y'(0) = 0, y(0) = 0$

We have $y' = -c_1 e^{-x} - 2c_2 e^{-2x}$

Thus, $y'(0) = 0$, $y(0) = 0$ are

$$0 = y'(0) = -c_1 e^{-0} - 2c_2 e^{-2(0)} = -c_1 - 2c_2$$

$$0 = y(0) = c_1 e^{-0} + c_2 e^{-2(0)} + 3 = c_1 + c_2 + 3$$

This gives

$$\boxed{\begin{aligned} -c_1 - 2c_2 &= 0 \\ c_1 + c_2 &= -3 \end{aligned}} \quad \begin{matrix} ① \\ ② \end{matrix}$$

① gives $c_1 = -2c_2$. Plug into ② to get $-2c_2 + c_2 = -3$. So, $c_2 = 3$. Thus, $c_1 = -2c_2 = -2(3) = -6$.

Thus, $c_1 = -6$, $c_2 = 3$.

So the solution is

$$\boxed{y = -6e^{-x} + 3e^{-2x} + 3}$$

3 (b) Solve

$$y'' + 2y' = 2x + 5 - e^{-2x}, \quad y'(0) = 1, \; y(0) = -1$$

In problem 2 we saw that the general solution to $y'' + 2y' = 2x + 5 - e^{-2x}$ is

$$y = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2} x e^{-2x}$$

Note that

$$y' = -2c_2 e^{-2x} + x + 2 + \frac{1}{2} e^{-2x} - x e^{-2x}$$

Thus, $y'(0) = 1$, $y(0) = -1$ gives

$$1 = y'(0) = -2c_2 e^{-2(0)} + (0) + 2 + \frac{1}{2} e^{-2(0)} - 0 \cdot e^{-2(0)}$$

$$-1 = y(0) = c_1 + c_2 e^{-2(0)} + \frac{1}{2}(0)^2 + 2(0) + \frac{1}{2}(0) e^{-2(0)}$$

This gives

$$\boxed{\begin{aligned} -2c_2 + 2 + \tfrac{1}{2} &= 1 \\ c_1 + c_2 &= -1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} -2c_2 &= -\tfrac{3}{2} \\ c_1 + c_2 &= -1 \end{aligned}} \begin{matrix} (1) \\ (2) \end{matrix}$$

(1) gives $c_2 = \frac{3}{4}$

(2) gives $c_1 = -1 - c_2 = -1 - \frac{3}{4} = -\frac{7}{4}$

Thus the solution is

$$\boxed{y = -\frac{7}{4} + \frac{3}{4} e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2} x e^{-2x}}$$